

# Preliminary Classification of Realizations of Two-Dimensional Lie Algebras of Vector Fields on a Circle

Stanislav V. SPICHAK

*Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str.,  
01601 Kyiv-4, Ukraine*  
E-mail: *stas\_sp@mail.ru*

Finite-dimensional subalgebras of a Lie algebra of smooth vector fields on a circle, as well as piecewise-smooth global transformations of a circle on itself, are considered. A canonical form of realizations of two-dimensional noncommutative algebra is obtained. It is shown that all other realizations of smooth vector fields are reduced to this form using global transformations.

## 1 Introduction

The description of Lie algebra representations by vector fields on a line and a plane was first considered by S. Lie [3, S. 1–121]. However, this problem is still of great interest and widely applicable. In spite to its importance for applications, only recently a complete description of realizations begun to be investigated systematically. Furthermore, only since the late eighties of the last century papers on that subject were published regularly. In particular, different problems of realizations were studied such as realizations of first order differential operators of a special form in [2], realizations of physical algebras (Galilei, Poincaré and Euclid ones) in [9, 11]. In [4] it was constructed a complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables. In that paper one can obtain a more complete review on the subject and a list of references.

Almost in all works on the subject realizations are considered up to local equivalence transformations. Attempts to classify realizations of Lie algebras in vector fields on some manifold with respect to global equivalence transformations (on the whole manifold) have been made only in a few papers (see, e.g., [6, 8, 10]). In these papers it is proved that (up to isomorphism) there are only three algebras, namely, one-dimensional, noncommutative two-dimensional and three-dimensional isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ , that can be realized by analytic vector fields on the circle.

The purpose of this paper is to construct all inequivalent realization of the two-dimensional algebras on the circle. For this reason, we are not limited by the requirement of analyticity, as it is considered in [6, 8, 10].

On the circle  $S^1$  we introduce the parameter  $\theta \in \mathbb{R}$ ,  $0 \leq \theta < 2\pi$ . Then the vector fields on  $S^1$  can be represented as a vector field  $v(\theta)\frac{d}{d\theta}$ , where  $v(\theta)$  is a smooth real function on the circle [5]. One more possibility is to assume that  $\theta \in \mathbb{R}$ , and  $v(\theta)$  is a smooth  $2\pi$ -periodic function on a line. We consider the vector fields of the class  $C^1$  (with continuously-differentiable functions  $v(\theta)$ ), which is natural to require when calculate the commutators of two vector fields.

We also introduce a class of transformations  $f: S^1 \rightarrow S^1$ , which is defined by the following properties:

- it is one-to-one mapping of the circle onto itself;
- $f(\theta)$  is continuous at any point  $\theta \in S^1$ ;
- it is continuously differentiable at all points except a finite number of them;
- the derivative  $f'(\theta)$  tends to  $-\infty$  or  $+\infty$  at all points of discontinuity;
- under a change of the coordinate  $\tilde{\theta} = f(\theta)$  a vector field of the class  $C^1$  transforms to a vector field of the same class.

Such transformations are defined as *equivalence transformations* of vector fields. We call two realizations of an algebra of vector fields inequivalent, if it is impossible to transform realizations to each other by compositions of equivalence transformations.

We assume that  $f(0) = 0$ , without loss of generality. Indeed, any equivalence transformation is obvious a composition of some equivalence transformations with fixed zero and a rotation of the circle. So, we can perform preliminary classification of inequivalent realizations up to such transformations (with fixed zero) and then complete classification taking into account the rotations of the circle. Then it is easy to see that  $f(\theta)$  is monotone on the whole interval  $0 \leq \theta < 2\pi$ . Degree of the map equals  $\pm 1$ ,  $\deg f = \pm 1$  (see [1]). Depending on the sign of the degree the function is monotonically decreasing or increasing.

Besides the transformations of  $f$ , we introduce the following class of *homotopy* of a circle into itself. Let us take two points  $\theta_1, \theta_2 \in S^1$ ,  $\theta_1 < \theta_2$ . We define a family of transformations  $F_{\theta_1, \theta_2}(t, \theta): [0, 1] \times S^1 \rightarrow S^1$  of the circle the as follows.

In the case  $\theta_1 \neq 0$

$$F_{\theta_1, \theta_2}(t, \theta) = \begin{cases} \theta + t \frac{\theta}{\theta_1} (\theta_2 - \theta_1) & \text{if } 0 \leq \theta < \theta_1, \\ \theta + t (\theta_2 - \theta) & \text{if } \theta_1 \leq \theta < \theta_2, \\ \theta & \text{if } \theta_2 \leq \theta < 2\pi. \end{cases}$$

In the case  $\theta_1 = 0$

$$F_{\theta_1, \theta_2}(t, \theta) = \begin{cases} \theta(1 - t) & \text{if } 0 \leq \theta < \theta_2, \\ \theta - t \frac{2\pi - \theta}{2\pi - \theta_2} \theta_2 & \text{if } \theta_2 \leq \theta < 2\pi. \end{cases}$$

Obviously, if some subset of singular points of a vector field (i.e., zeros of its coefficient [1]) forms an open set  $(\theta_1, \theta_2)$  on  $S^1$  then this interval can be constricted

to a point by an appropriate homotopy. Therefore, we can consider vector fields singularities which do not form intervals, although the number of singularities may be infinite. We denote the class of such vector fields by  $\mathcal{C}$ . They form the set of realizations of the one-dimensional algebra. In general case such vector fields cannot be simplified by equivalence transformations, because of infinity the number of singular points.

We are interested in all inequivalent realizations of finite-dimensional Lie algebras by vector fields from the class  $\mathcal{C}$ .

## 2 Two-dimensional commutative algebra

Further we denote vector fields  $v(\theta)\frac{d}{d\theta}$  and  $w(\theta)\frac{d}{d\theta}$  by  $V$  and  $W$ , correspondingly. Let them commute, and  $V \in \mathcal{C}$ . A singular point  $\theta_0$  of the field  $V$  is said to be *degenerate* if  $v'(\theta_0) = 0$ . It is easily to show that if  $0 \leq \theta_0 < \theta_1 < 2\pi$  are two degenerate points, such that the interval  $(\theta_0, \theta_1)$  does not contain additional degenerate points, then  $w(\theta) = \lambda v(\theta)$  on  $(\theta_0, \theta_1)$ , where  $\lambda \neq 0$  is an arbitrary constant. This follows from the fact that  $W \in \mathcal{C}$  and the continuity of the derivative of the function  $w(\theta)$ . In particular, if there is no degenerate point, or it is unique, then  $w(\theta) = \lambda v(\theta)$  on  $S^1$ . In this case the vector fields  $V$  and  $W$  are linearly dependent, and there is no realization of two-dimensional commutative Lie algebra (see [6, 8, 10]).

We assume that we have more than one degenerate point. Without loss of generality, we may assume that point 0 (and  $2\pi$ ) is degenerate. Then the function  $w(\theta)$  can be described as follows. We take an arbitrary point  $\theta \in S^1$ . If it is non-degenerate for function  $v(\theta)$ , then, obviously, there is a maximum interval  $(\theta_0, \theta_1)$  with two degenerate endpoints on it, such that  $0 \leq \theta_0 < \theta < \theta_1 < 2\pi$ . Then  $w(\theta) = \lambda v(\theta)$  on this interval. Further, considering point  $\theta' \notin [\theta_0, \theta_1]$ , we repeat the procedure, if it is not degenerate. Again we have relation  $w(\theta) = \lambda' v(\theta)$  on some interval. Here the values  $\lambda$  and  $\lambda'$  can be not equal.

If the point  $\theta'$  is degenerate, then, by virtue of the fact that  $V \in \mathcal{C}$ , we can find arbitrarily close to it a non-degenerate point  $\theta''$ . So we repeat the above procedure for  $\theta''$ . Thus if we know the function  $v(\theta)$ , we can construct values of the function  $w(\theta)$  in all non-degenerate points. For degenerate points we can construct values of  $w(\theta)$  in arbitrary close to them points. If the number of degenerate points of the function  $v(\theta)$  is infinite, then the number of non-equivalent realizations of the vector field  $W$  is infinite. So the number of realizations of two-dimensional commutative algebra is infinite.

## 3 Two-dimensional noncommutative algebra.

### Auxiliary lemmas

Let vector fields  $V$  and  $W$  generate the noncommutative algebra. It can be assumed up to their linear combining that they satisfy the commutation relation

$[V, W] = W$ , which implies the relation between the functions  $v(\theta)$  and  $w(\theta)$ :

$$v(\theta)w'(\theta) - v'(\theta)w(\theta) = w(\theta). \quad (1)$$

**Lemma 1.** *There is a singular point for the field  $W$ .*

*Proof.* Assume that the field  $W$  has no singular points, i.e.,  $w(\theta) > 0$  (or  $w(\theta) < 0$ ) for all values  $0 \leq \theta < 2\pi$ . Then from (1) we can obtain the solution for the function  $v(\theta)$  on the whole interval  $[0, 2\pi)$ :

$$v(\theta) = \left( - \int_0^\theta \frac{d\vartheta}{w(\vartheta)} + \lambda \right) w(\theta), \quad (2)$$

where  $\lambda$  is some constant. The function  $w(\theta)$  is  $2\pi$ -periodic on the line. Since the integrand in (2) is positive, we have  $v(0) \neq v(2\pi)$ . It contradicts the periodicity of the function  $v(\theta)$ .  $\square$

**Lemma 2.** *Singular points of  $W$  are singular points of  $V$ .*

*Proof.* Assume that  $w(\theta_0) = 0$ . Suppose that  $v(\theta_0) \neq 0$ . Then there exists some neighborhood  $U_{\theta_0}$  of this point, where  $v(\theta) \neq 0$ . In this neighborhood, the equation (1) can be rewritten as:

$$w'(\theta) = \frac{1 + v'(\theta)}{v(\theta)} w(\theta). \quad (3)$$

Since  $w(\theta_0) = 0$  and the right-hand side of equation (3) satisfies the Lipschitz condition with respect to  $w$  uniformly on  $\theta$  then, by virtue of Picard's theorem [7], the differential equation (3) has a unique solution in the neighborhood of  $\theta_0$ . Obviously, such a solution is  $w \equiv 0$ , what contradicts the fact that  $W \in \mathcal{C}$ .  $\square$

**Lemma 3.** *The number of singular points of the field  $W$  is finite.*

*Proof.* Assume that the number of singular points is infinite. Since  $S^1$  is a compact set then there is a monotonically increasing (or decreasing) a sequence  $\{\theta_n\}$  converging to some point  $\theta_0$  such that  $w(\theta_n) = 0$ . It is easy to show that for any  $n$  there is a non-singular point  $\hat{\theta}_n \in (\theta_n, \theta_{n+1})$  satisfying the condition  $w'(\hat{\theta}_n) = 0$ . Then, from equation (1) we see that  $v'(\hat{\theta}_n) = -1$ . Since  $\lim_{n \rightarrow \infty} \hat{\theta}_n = \theta_0$ , then, by virtue of continuous differentiability of function  $v(\theta)$ , we have  $\lim_{n \rightarrow \infty} v'(\hat{\theta}_n) = v'(\theta_0) = -1$ . On the other hand, from Lemma 2 we get that  $v(\theta_n) = v(\theta_{n+1}) = 0$ . Hence there is a point  $\tilde{\theta}_n \in (\theta_n, \theta_{n+1})$  such that  $v'(\tilde{\theta}_n) = 0$ . And since  $\lim_{n \rightarrow \infty} \tilde{\theta}_n = \theta_0$ , then  $\lim_{n \rightarrow \infty} v'(\tilde{\theta}_n) = v'(\theta_0) = 0$ . Therefore, we have a contradiction.  $\square$

**Lemma 4.** *If  $\theta_0$  is a singular point of  $W$ , then it is degenerate for this field (i.e.,  $w'(\theta_0) = 0$ ).*

*Proof.* Since  $v(\theta_0) = 0$  (see Lemma 2), then

$$v(\theta) = v'(\theta_0)(\theta - \theta_0) + h(\theta), \quad w(\theta) = w'(\theta_0)(\theta - \theta_0) + g(\theta),$$

where  $h, g$  are continuously differentiable functions and

$$h(\theta_0) = g(\theta_0) = h'(\theta_0) = g'(\theta_0) = 0. \quad (4)$$

Furthermore, taking to account equation (1) we have

$$\begin{aligned} \left[ v(\theta) \frac{d}{d\theta}, w(\theta) \frac{d}{d\theta} \right] &= [(\theta - \theta_0)(v'(\theta_0)g'(\theta) - w'(\theta_0)h'(\theta)) \\ &\quad + h(\theta)(w'(\theta_0) + g'(\theta)) - g(\theta)(v'(\theta_0) + h'(\theta))] \frac{d}{d\theta} \\ &= [w'(\theta_0)(\theta - \theta_0) + g(\theta)] \frac{d}{d\theta}. \end{aligned}$$

Let us divide both sides of this equality by  $\theta - \theta_0$  and take the limit  $\theta \rightarrow \theta_0$ . Then from relations (4) and L'Hopital theorem one can easily see that  $w'(\theta_0) = 0$ .  $\square$

**Lemma 5.** *Under the equivalence transformation  $\tilde{\theta} = f(\theta)$  singular (resp. regular) points of the vector field  $W$  are mapped to singular (resp. regular) points of the vector field  $\tilde{W} = \tilde{w}(\tilde{\theta}) \frac{d}{d\tilde{\theta}}$ .*

*Proof.* a) Let  $w(\theta_0) \neq 0$ . Then there is a finite derivative  $f'(\theta_0)$ . If it is not, then there is a neighborhood of the singular point  $\theta_0$ , where it is continuous and  $\lim_{\theta \rightarrow \theta_0} f'(\theta) = \pm\infty$  (the sign depends on  $\deg f$ ). For the transformed vector field  $\tilde{W}$  we have the relation  $\tilde{w}(\tilde{\theta}) \frac{d}{d\tilde{\theta}} = w(\theta) f'(\theta) \frac{d}{d\theta}$ . Hence,  $\tilde{w}(f(\theta)) = w(\theta) f'(\theta)$ . Since  $w(\theta) \neq 0$  in the above neighborhood, then

$$f'(\theta) = \frac{\tilde{w}(f(\theta))}{w(\theta)}, \quad (5)$$

and from the continuity of the functions  $f$  and  $\tilde{w}$  it is following that the limit  $\lim_{\theta \rightarrow \theta_0} f'(\theta)$  is finite. Recall that a continuity of the function  $\tilde{w}$  follows from the properties that any equivalence transformation  $f$  maps  $C^1$ -vector fields to  $C^1$ -vector fields.

Now suppose that  $\tilde{w}(f(\theta_0)) = 0$  (i.e.,  $f(\theta_0)$  is singular). Since the right side of differential equation (5) satisfies Lipschitz condition for the argument  $f(\theta)$  uniformly on  $\theta$  (function  $\tilde{w}$  is continuously differentiable), then in a neighborhood of the point  $\theta_0$  a unique solution  $f(\theta)$  exists. The constant solution in this neighborhood  $f(\theta) = f(\theta_0) = \text{const}$  satisfies equation (5). But the definition of the equivalence transformation contradicts to the property of one-to-one mapping. That is  $\tilde{\theta}_0 = f(\theta_0)$  is a regular point.

b) Let  $\theta_0$  be a singular point and suppose that  $\tilde{w}(f(\theta_0)) \neq 0$  (i.e.,  $f(\theta_0)$  is regular). By Lemma 3 there is a neighborhood of  $\theta_0$ , in which all points are

regular (except  $\theta_0$ ). From the previous part of the proof we get (see (5)) for the regular points relations  $f'(\theta) > 0$  ( $f'(\theta) < 0$ ) if  $\deg f > 0$  ( $\deg f < 0$ ). Therefore, it is easy to show that the inverse map  $\theta = f^{-1}(\tilde{\theta})$  belongs to the class of equivalence transformations. Applying the reasoning of the previous part, we see that the regular point  $\tilde{\theta}_0$  goes to the regular point  $\theta_0$  under the mapping  $f^{-1}$ . So we have a contradiction.  $\square$

**Corollary 1.** *The number of singular points is an invariant of any equivalence transformation.*

## 4 Realizations of a two-dimensional noncommutative algebra

Taking into account Lemmas 1 and 3, we suppose that there is a vector field  $W$  with  $n \geq 1$  singular points  $\theta_k$ . It is easy to show that applying the composition of equivalence transformations and rotation of the circle we achieve that  $\theta_k = \frac{2\pi k}{n}$ ,  $k = 0, 1, \dots, n-1$ . Consider the interval  $\Delta_k = (\theta_k, \theta_{k+1})$  and denote

$$\bar{\theta}_k = \frac{\theta_k + \theta_{k+1}}{2} = \frac{\pi(2k+1)}{n}.$$

We construct the following continuously differentiable transformation for  $f$  on  $\Delta_k$  satisfying the conditions

$$f(\theta_k) = \theta_k, \quad f(\theta_{k+1}) = \theta_{k+1}, \quad f(\bar{\theta}_k) = \bar{\theta}_k. \quad (6)$$

Suppose that  $w(\theta) > 0$ ,  $\theta \in \Delta_k$ . Consider the Cauchy problem for this interval:

$$w(\theta)f'(\theta) = 1 - \cos(nf(\theta)), \quad f(\bar{\theta}_k) = \bar{\theta}_k. \quad (7)$$

Its solution is

$$f(\theta) = \frac{2}{n} \arctan(-nI(\theta)) + \theta_k, \quad \text{where} \quad I(\theta) = \int_{\bar{\theta}_k}^{\theta} \frac{d\theta}{w(\theta)}. \quad (8)$$

The integral  $I(\theta)$  converges for any point of the interval  $\Delta_k$ . By virtue of Lemma 4 the integral diverges at the ends of this interval:

$$\lim_{\theta \rightarrow \theta_k + 0} I(\theta) = -\infty, \quad \lim_{\theta \rightarrow \theta_{k+1} - 0} I(\theta) = +\infty.$$

It is easy to show that the transformation (8) satisfies conditions (6) and maps the vector field  $w(\theta)\frac{d}{d\theta}$  to the vector field  $(1 - \cos(n\tilde{\theta}))\frac{d}{d\tilde{\theta}}$ .

If  $w(\theta) < 0$  for  $\theta \in \Delta_k$  then we can analogously obtain the equivalence transformation that maps the vector field  $w(\theta)\frac{d}{d\theta}$  to  $(\cos(n\tilde{\theta}) - 1)\frac{d}{d\tilde{\theta}}$ .

Now, if we consider the vector field at the intervals  $\Delta_k$ ,  $k = 0, 1, \dots, n-1$ , then substituting the function  $w(\theta) = \pm(\cos(n\theta) - 1)$  (omitting the tilde) in

equation (1), it is easy to obtain the solution for the function  $v(\theta)$  on these specified intervals:

$$v(\theta) = \frac{1}{n} \sin(n\theta) + \lambda_k(1 - \cos(n\theta)), \quad \lambda_k \in \mathbb{R}. \quad (9)$$

As a result, we have the following assertion.

**Theorem 1.** *Any realization of the two-dimensional noncommutative algebra of vector fields on a circle is equivalent to the form*

$$\left\langle \left( \frac{1}{n} \sin(n\theta) + \lambda_k(\theta)(1 - \cos(n\theta)) \right) \frac{d}{d\theta}, \sigma_k(\theta)(1 - \cos(n\theta)) \frac{d}{d\theta} \right\rangle, \quad (10)$$

where  $\lambda_k(\theta)$  and  $\sigma_k(\theta) = \pm 1$  are constants on the intervals  $\left( \frac{2\pi k}{n}, \frac{2\pi(k+1)}{n} \right)$ .

The questions about the reducibility of realizations (10) to simpler ones, their inequivalence and the number of inequivalent realizations will be discussed in a forthcoming work.

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